FP2 2008 Adapted

1. Solve the differential equation $\frac{dy}{dx} - 3y = x$

to obtain y as a function of x. (Total 5 marks)

$\text{IF } f(x) = e^{-\int 3dx} = e^{-3x} \Rightarrow e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = xe^{-3x}$

$\Rightarrow \frac{d}{dx}(ye^{-3x}) = xe^{-3x} \Rightarrow ye^{-3x} = \int xe^{-3x}dx$ $\quad u = x \quad v = -\frac{1}{3}e^{-3x}$

$u' = 1 \quad v' = e^{-3x}$

$\Rightarrow ye^{-3x} = -\frac{1}{3}xe^{-3x} + \frac{1}{3}\int e^{-3x}dx$

$\Rightarrow ye^{-3x} = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + c \quad \therefore y = -\frac{1}{3}x - \frac{1}{9} + ce^{3x}$

2. (a) Simplify the expression $\frac{(x+3)(x+9)}{x-1} - (3x-5)$, giving your answer in the form $\frac{a(x+b)(x+c)}{x-1}$, where a, b and c are integers. (4)

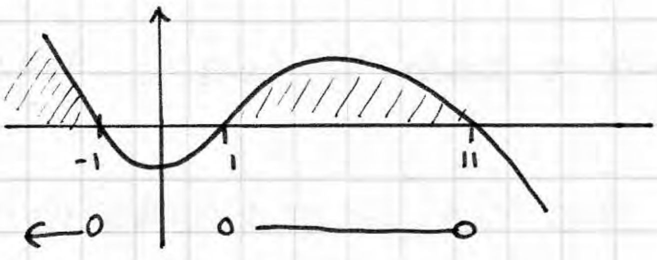
(b) Hence, or otherwise, solve the inequality $\frac{(x+3)(x+9)}{x-1} > 3x - 5$ **(4)(Total 8 marks)**

$\frac{(x+3)(x+9) - (3x-5)(x-1)}{(x-1)} = \frac{x^2+12x+27 - 3x^2+8x-5}{(x-1)}$

$\Rightarrow \frac{-2x^2+20x+22}{(x-1)} = \frac{-2(x^2-10x-11)}{(x-1)} = \frac{-2(x-11)(x+1)}{(x-1)}$

b) $\frac{-2(x-11)(x+1)}{(x-1)} > 0 \Rightarrow -2(x-11)(x+1)(x-1) > 0$

11   -1   1



$x < -1$

or

$1 < x < 11$

3. (a) Find the general solution of the differential equation $3\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = x^2$ **(8)**

(b) Find the particular solution for which, at $x = 0$, $y = 2$ and $\frac{dy}{dx} = 3$.(6)(Total 14 marks)

$y = Ae^{mx}$
$y' = Ame^{mx}$
$y'' = Am^2e^{mx}$

$3y'' - y' - 2y = 0$

$\underset{\neq 0}{Ae^{mx}}\underset{=0}{(3m^2 - m - 2)} = 0$

$(3m+2)(m-1) = 0$

$m = -\frac{2}{3} \quad m = 1$

$y_{cf} = Ae^{x} + Be^{-\frac{2}{3}x}$

$y_{PI} = -\frac{1}{2}x^2 + \frac{1}{2}x - \frac{7}{4}$

$\therefore \quad y = Ae^{x} + Be^{-\frac{2}{3}x} - \frac{1}{2}x^2 + \frac{1}{2}x - \frac{7}{4}$

$y = ax^2 + bx + c$
$y' = 2ax + b$
$y'' = 2a$

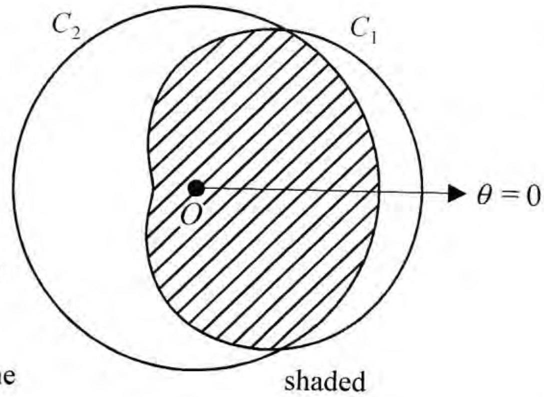
$3y'' = 6a$
$-y' = -b - 2ax$
$-2y = -2c - 2bx - 2ax^2$

$x^2 = (6a - b - 2c) - (2a + 2b)x - 2ax^2$

$\therefore a = -\frac{1}{2} \quad b = \frac{1}{2} \quad -3 - \frac{1}{2} - 2c = 0$

$c = -\frac{7}{4}$

4. The diagram above shows the curve C_1 which has polar equation $r = a(3 + 2\cos\theta)$, $0 \le \theta < 2\pi$ and the circle C_2 with equation $r = 4a$, $0 \le \theta < 2\pi$, where a is a positive constant.



(a) Find, in terms of a, the polar coordinates of the points where the curve C_1 meets the circle C_2.(4)

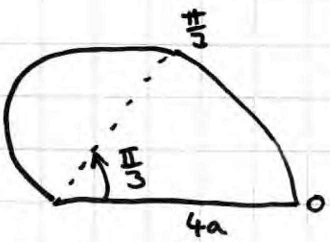
The regions enclosed by the curves C_1 and C_2 overlap and this common region R is shaded in the figure.

(b) Find, in terms of a, an exact expression for the area of the region R.(8)

(c) In a single diagram, copy the two curves in the diagram above and also sketch the curve C_3 with polar equation $r = 2a\cos\theta$, $0 \le \theta < 2\pi$ Show clearly the coordinates of the points of intersection of C_1, C_2 and C_3 with the initial line, $\theta = 0$.(3)(Total 15 marks)

a) $a(3 + 2\cos\theta) = 4a \Rightarrow 2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2} \quad \theta = \frac{\pi}{3}, -\frac{\pi}{3} \quad (4a, \frac{\pi}{3}); (4a, -\frac{\pi}{3})$

b)



Area $= 2 \times [\text{Sector} + \text{Cardioid part}]$

$= 2[\frac{1}{2}(4a)^2\frac{\pi}{3} + \frac{1}{2}a^2\int_{\frac{\pi}{3}}^{\pi}(3 + 2\cos\theta)^2 d\theta]$

$= a^2[\frac{16\pi}{3} + \int_{\frac{\pi}{3}}^{\pi} 9 + 12\cos\theta + 4\cos^2\theta \, d\theta]$

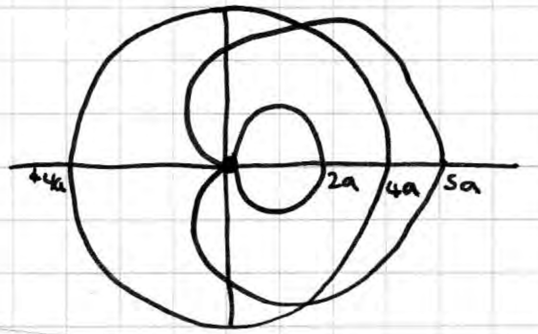
$\cos 2\theta = 2\cos^2\theta - 1$
$4\cos^2\theta = 2\cos 2\theta + 2$

$= a^2[\frac{16\pi}{3} + \int_{\frac{\pi}{3}}^{\pi} 11 + 12\cos\theta + 2\cos 2\theta \, d\theta]$

$= a^2(\frac{16\pi}{3} + [11\theta + 12\sin\theta + \sin 2\theta]_{\frac{\pi}{3}}^{\pi})$

$$= a^2\left(\frac{16\pi}{3} + \left[(11\pi) - \left(\frac{11\pi}{3} + 6\sqrt{3} + \frac{\sqrt{3}}{2}\right)\right]\right) = a^2\left(\frac{16\pi}{3} + \frac{22\pi}{3} - \frac{13\sqrt{3}}{2}\right) = \frac{a}{6}$$

c)



5. (a) Find, in terms of k, the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = kt + 5\text{, where } k \text{ is a constant and } t > 0.(7)$$

For large values of t, this general solution may be approximated by a linear function.

(b) Given that $k = 6$, find the equation of this linear function.(2)(Total 9 marks)

$x = Ae^{mt}$

$x' = Ame^{mt}$

$x'' = Am^2e^{mt}$

$x'' + 4x' + 3x = 0$

$Ae^{mt}(m^2 + 4m + 3) = 0$

$\neq 0 \quad = 0$

$(m+3)(m+1) = 0$

$m = -3 \quad m = -1$

$\therefore x_{cf} = Ae^{-t} + Be^{-3t}$

$x = at + b$

$x' = a$

$x'' = 0$

$x'' = 0$

$+ \ 4x' = 4a$

$+ \ 3x = 3at + 3b$

$kt + 5 \equiv 3at + 4a + 3b$

$\therefore a = \frac{1}{3}k \quad \frac{4}{3}k + 3b = 5$

$\Rightarrow b = -\frac{4}{9}k + \frac{5}{3}$

$\Rightarrow x_{PI} = \frac{1}{3}kt - \frac{4}{9}k + \frac{5}{3}$

$\therefore x = Ae^{-t} + Be^{-3t} + \frac{1}{3}kt - \frac{4}{9}k + \frac{5}{3}$

b) as $t \to \infty \quad Ae^{-t} + Be^{-3t} \to 0 \quad x \to 2t - 1$

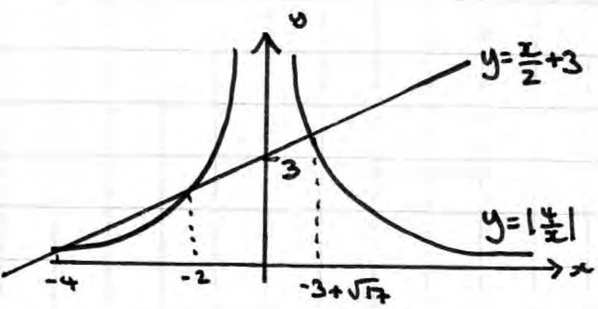
6. (a) Find, in the simplest surd form where appropriate, the exact values of x for which

$$\frac{x}{2}+3=\left|\frac{4}{x}\right|. \quad (5)$$

(b) Sketch, on the same axes, the line with equation $y=\frac{x}{2}+3$ and the graph of

$y=\left|\frac{4}{x}\right|$, $x\neq 0$. (3)

(c) Find the set of values of x for which $\frac{x}{2}+3>\left|\frac{4}{x}\right|$. (2)(Total 10 marks)



$\frac{4}{x}=\frac{x}{2}+3$

$8=x^2+6x$

$x^2+6x-8=0$

$(x+3)^2=17$

$x=-3\pm\sqrt{17}$

$\frac{4}{x}=-\frac{x}{2}-3$

$8=-x^2-6x$

$x^2+6x+8=0$

$(x+4)(x+2)=0$

$x=-4 \quad x=-2$

$-4<x<-2$ or $x>-3+\sqrt{17}$

2

7. (a) Show that the substitution $y=vx$ transforms the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x}{y}+\frac{3y}{x}, \quad x>0,\ y>0 \qquad \text{(I)}$$

into the differential equation $\quad x\frac{\mathrm{d}v}{\mathrm{d}x}=2v+\frac{1}{v}.$ (II) (3)

(b) By solving differential equation (II), find a general solution of differential equation (I) in the form $y=\mathrm{f}(x)$. (7)

Given that $y=3$ at $x=1$, (c) find the particular solution of differential equation (I). (2)

$y=vx$

$\frac{dy}{dx}=x\frac{dv}{dx}+v \qquad \frac{d^2y}{dx^2}=x\frac{d^2v}{dx^2}+2\frac{dv}{dx}$

$\Rightarrow x\frac{dv}{dx}+v=\frac{x}{vx}+\frac{3vx}{x} \Rightarrow x\frac{dv}{dx}+v=\frac{1}{v}+3v \Rightarrow x\frac{dv}{dx}=\frac{1}{v}+2v$

$\Rightarrow x\frac{dv}{dx}=\frac{2v^2+1}{v} \Rightarrow \int\frac{\frac{1}{4}(4v)}{2v^2+1}\,dv=\int\frac{1}{x}\,dx$

$\Rightarrow \frac{1}{4}\ln|2v^2+1|=\ln|x|+c$

$\Rightarrow \ln|2v^2+1|=\ln x^4+4c \Rightarrow 2v^2+1=Ax^4 \qquad A=e^{4c}$

$\Rightarrow v=\sqrt{\frac{Ax^4-1}{2}} \Rightarrow v=\sqrt{Bx^4-\frac{1}{2}} \qquad B=\frac{A}{2}$

$\Rightarrow \frac{y^2}{x^2}=Bx^4-\frac{1}{2} \Rightarrow y=\sqrt{Bx^6-\frac{1}{2}x^2}$

c) $(1,3)$

$9=B-\frac{1}{2} \therefore B=\frac{19}{2}$

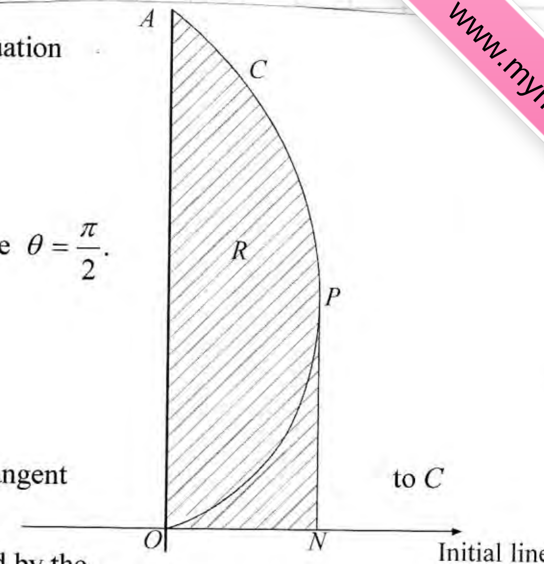
$\therefore y=\sqrt{\frac{19}{2}x^6-\frac{1}{2}x^2}$

8. The curve C shown in the diagram above has polar equation

$$r = 4(1-\cos\theta),\ 0 \le \theta \le \frac{\pi}{2}.$$

At the point P on C, the tangent to C is parallel to the line $\theta = \frac{\pi}{2}$.

(a) Show that P has polar coordinates $\left(2, \frac{\pi}{3}\right)$.(5)



The curve C meets the line $\theta = \frac{\pi}{2}$ at the point A. The tangent to C at P meets the initial line at the point N. The finite region R, shown shaded in the diagram above, is bounded by the initial line, the line $\theta = \frac{\pi}{2}$, the arc AP of C and the line PN.

(b) Calculate the exact area of R.

(8)

parallel to $\theta = \frac{\pi}{2} \Rightarrow \frac{dx}{d\theta} = 0$ $\quad x = r\cos\theta = 4(1-\cos\theta)\cos\theta$

$\Rightarrow x = 4\cos\theta - 4\cos^2\theta$

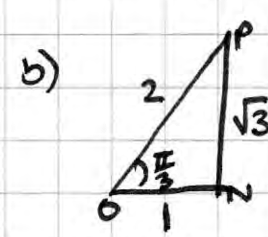
$\Rightarrow \frac{dx}{d\theta} = -4\sin\theta + 8\cos\theta\sin\theta$

$\Rightarrow \quad 8\cos\theta\,\cancel{\sin\theta} = 4\,\cancel{\sin\theta} \Rightarrow \cos\theta = \frac{1}{2}$

$\therefore \theta = \frac{\pi}{3}$

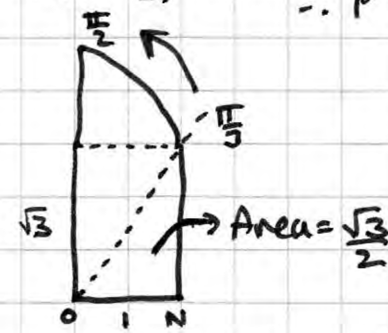
$r = 4(1-\cos\theta)$ $\quad \theta = \frac{\pi}{3}$ $\quad r = 4(1-\frac{1}{2}) = 2$ $\quad \therefore P(2, \frac{\pi}{3})$ #

b)



$\Rightarrow ON = 1$

$NP = \sqrt{3}$



$\therefore R = \frac{\sqrt{3}}{2} + \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 16(1-\cos\theta)^2\,d\theta$

$R = \frac{\sqrt{3}}{2} + 8\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 1 - 2\cos\theta + (\frac{1}{2}\cos 2\theta + \frac{1}{2})\,d\theta = \frac{\sqrt{3}}{2} + 8\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\,d\theta$

$R = \frac{\sqrt{3}}{2} + 4\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 3 - 4\cos\theta + \cos 2\theta\,d\theta = \frac{\sqrt{3}}{2} + 4\left[3\theta - 4\sin\theta + \frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$

$R = \frac{\sqrt{3}}{2} + 4\left[\left(\frac{3\pi}{2} - 4\right) - \left(\frac{3\pi}{3} - 2\sqrt{3} + \frac{\sqrt{3}}{4}\right)\right] = \frac{\sqrt{3}}{2} + 4\left[\frac{\pi}{2} + \frac{7\sqrt{3}}{4} - 4\right]$

$R = \frac{\sqrt{3}}{2} + 2\pi + 7\sqrt{3} - 16 \quad \therefore R = 2\pi + \frac{15\sqrt{3}}{2} - 16.$

9 [redacted].

$$(x^2+1)\frac{d^2y}{dx^2} = 2y^2 + (1-2x)\frac{dy}{dx} \qquad \text{(I)}$$

(a) By differentiating equation (I) with respect to x, show that

$$(x^2+1)\frac{d^3y}{dx^3} = (1-4x)\frac{d^2y}{dx^2} + (4y-2)\frac{dy}{dx}. \qquad (3)$$

Given that $y = 1$ and $\frac{dy}{dx} = 1$ at $x = 0$,

(b) find the series solution for y, in ascending powers of x, up to and including the term in x_3.(4)

(c) Use your series to estimate the value of y at $x = -0.5$, giving your answer to two decimal places.(1)

$$\frac{d}{dx}\left[(x^2+1)\frac{d^2y}{dx^2}\right] = \frac{d}{dx}(2y^2) + \frac{d}{dx}\left[(1-2x)\frac{dy}{dx}\right]$$

$$= 2x\frac{d^2y}{dx^2} + (x^2+1)\frac{d^3y}{dx^3} = 4y\frac{dy}{dx} - 2\frac{dy}{dx} + (1-2x)\frac{d^2y}{dx^2}$$

$$\Rightarrow (x^2+1)\frac{d^3y}{dx^3} = (1-4x)\frac{d^2y}{dx^2} + (4y-2)\frac{dy}{dx}$$

$x_0 = 0 \quad y_0 = 1 \quad y_0' = 1 \quad \Rightarrow (1)y_0'' = 2(1)^2 + (1-2(0))y' \Rightarrow y_0'' = 2+1 = 3$

$\Rightarrow (1)y_0''' = (3) + 2(1) \qquad \Rightarrow y_0''' = 5$

$\therefore \quad y = 1 + x + \frac{3}{2}x^2 + \frac{5}{6}x^3 \qquad x = -0.5 \Rightarrow y \approx 0.77$

10 The point P represents a complex number z on an Argand diagram such that

$$|z-3| = 2\,|z|.$$

(a) Show that, as z varies, the locus of P is a circle, and give the coordinates of the centre and the radius of the circle.(5)

The point Q represents a complex number z on an Argand diagram such that

$$|z+3| = |z - i\sqrt{3}|.$$

(b) Sketch, on the same Argand diagram, the locus of P and the locus of Q as z varies.(5)

(c) On your diagram shade the region which satisfies

$$|z-3| \geq 2\,|z| \text{ and } |z+3| \geq |z - i\sqrt{3}|. \qquad (2)$$

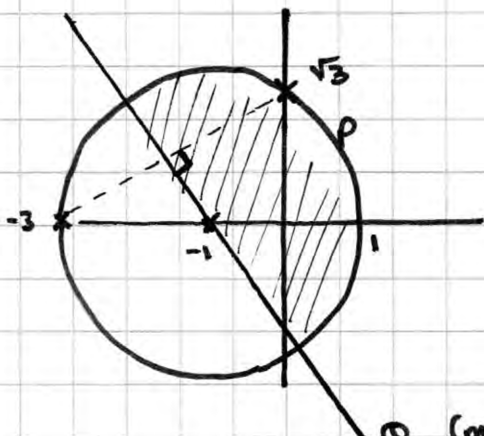
$|(x-3)+iy| = 2|x+iy| \Rightarrow (x-3)^2+y^2 = 4x^2+4y^2$

$\Rightarrow x^2-6x+9+y^2 = 4x^2+4y^2 \Rightarrow 3x^2+3y^2+6x = 9$

$\Rightarrow x^2+2x+y^2 = 3 \Rightarrow (x+1)^2+y^2 = 2^2$

locus is a circle $C(-1,0)$ $r=2$

$x=0 \Rightarrow y^2=3 \Rightarrow y=\sqrt{3}$



$|z-3| \geq 2|z|$

$\Rightarrow$ within circle

$|z+3| \geq |z-i\sqrt{3}|$

$\Rightarrow$ closer to $i\sqrt{3}$

Q (must pass through centre)

12. De Moivre's theorem states that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for $n \in \Re$

(a) Use induction to prove de Moivre's theorem for $n \in \mathbb{Z}^+$. (5)

(b) Show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ (5)

(c) Hence show that $2\cos\frac{\pi}{10}$ is a root of the equation

$$x^4 - 5x^2 + 5 = 0$$

$n=1$ $(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$ $\quad \cos n\theta + i\sin n\theta = \cos\theta + i\sin\theta$

$\therefore$ true when $n=1$

assume true if $n=k$ $\Rightarrow (\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

$n=k+1$ $(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$

$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$

$= \cos(k\theta + \theta) + i\sin(k\theta + \theta) = \cos(k+1)\theta + i\sin(k+1)\theta$

$\Rightarrow (\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta$

$\therefore$ true for $n=1$ and true for $n=k+1$ if true for $n=k$

$\therefore$ by Mathematical Induction true for $n \in \mathbb{Z}^+$

b) $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$(\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$

equating real parts $\Rightarrow \cos 5\theta \equiv \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$\therefore \cos 5\theta = \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1 - 2\cos^2\theta + \cos^4\theta)$

$\therefore \cos 5\theta = \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$

$\therefore \cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ #

c) $x = 2\cos\theta \Rightarrow x^4 - 5x^2 + 5 = 16\cos^4\theta - 20\cos^2\theta + 5$

$\Rightarrow x^4 - 5x^2 + 5 = \frac{\cos 5\theta}{\cos\theta} = 0 \Rightarrow \cos 5\theta = 0 \Rightarrow 5\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$\therefore \theta = \frac{\pi}{10}, \frac{3\pi}{10}, \ldots$

$\therefore x = 2\cos\frac{\pi}{10}$ must be a root of the equation $x^4 - 5x^2 + 5 = 0$